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UPPER BOUNDS FOR GROWTH IN THE RYLL–NARDZEWSKI FUNCTION OF AN ω-CATEGORICAL, ω-STABLE THEORY

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ABSTRACT

We establish a sharp upper bound for growth in the sequence $s_k(T) :=$ the number of k-types consistent with T, for T ω-categorical and ω-stable.

Introduction

Let T be a complete ω-categorical, ω-stable theory in a countable language. For each positive integer k, let $s_k(T)$ denote the number of complete k-types consistent with T. By the Ryll–Nardzewski Theorem, $s_k(T)$ is finite for all k and equals the number of Aut$\mathcal{M}$-orbits on M^k, where $\mathcal{M}$ is the countable model of T. In general, the sequence $s_k(T)$ can grow arbitrarily fast [C]. In section 1 we establish a sharp upper bound for the growth in $s_k(T)$ for ω-stable, ω-categorical T. Our proof uses the structure theory for ω-categorical, ω-stable structures developed by Cherlin, Harrington and Lachlan in [CHL]. Intuitively speaking, what we show is that the process of building a strucure of positive rank from strictly minimal components cannot increase the "order" of the growth rate. Specifically, it follows from our results that given any ω-categorical, ω-stable theory T, there is a strictly minimal theory T' such that $s_k(T')$ eventually dominates $s_k(T)$.

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An example due to E. Hrushovski shows that the hypothesis of ω-stability cannot be weakened to just stability.

In section 2 we apply the methods developed in the proof of the main Theorem (1.1) to bound the growth in

$$a_k := \max\{|\mathrm{acl}(A)| : |A| = k, A \subseteq M, \mathcal{M} \models T\},$$

and in section 3, we obtain a more restrictive bound for $s_k(T)$ when T satisfies an additional hypothesis.

Our notation generally follows [CHL]. Throughout $\mathcal{M}$ is a countable model of a theory T for the language L. We assume that L contains no function symbols. For the purpose of counting types, this assumption is harmless, since for any ω-categorical theory T, there is a purely relational theory T' such that for all $k, s_k(T') = s_k(T)$. If $A \subseteq M$ then $\mathcal{M}_A$ stands for the canonical expansion of $\mathcal{M}$ to a structure for the language $L(A)$ which adds constants for the elements of $A, \mathcal{M}|A = \langle A, R_\varphi^{\mathcal{M}|A}\rangle_{\varphi \in L}$ with $R_\varphi^{\mathcal{M}|A} = \{\bar{a} \in A : \mathcal{M} \models \varphi(\bar{a})\}, S_n(A)$ denotes the set of all complete $L(A)$-n-types consistent with $\mathrm{Th}(\mathcal{M}_A), S(A) = \cup_{n\in\omega} S_n(A)$ and $S(A, \mathcal{M})$ consists of all types in $S(A)$ that are realized in $\mathcal{M}$. We write $S(T)$ for $S(\emptyset)$. Following [CHL], we define an **extension by definitions** of $\mathcal{M}$ as follows. If E is a 0-definable equivalence relation on M^n for some $n \in \omega$, then the **E-extension** of $\mathcal{M}$ is the structure $\mathcal{M}^*$ over the language $L^* = L \cup \{U, V\}$ with universe $M \cup M^n/E$, and interpretations $U^{\mathcal{M}^*} = M, V^{\mathcal{M}^*} = \{(\bar{a}, \bar{a}/E) : \bar{a} \in M^n\}, R^{\mathcal{M}^*} = R^{\mathcal{M}}$, and $c^{\mathcal{M}^*} = c^{\mathcal{M}}$ for all relation symbols R and constant symbols c in L. Type always means complete type and rank and degree, abbreviated (rk, deg), always means Morley rank and degree. We write $s_k(A, B)$ for the number of k-types over A realized by elements of B in $\mathcal{M}$.

1. Bounding $s_k(T)$

The main result of this paper is the following:

THEOREM 1.1: *Let T be ω-categorical and ω-stable. Then there is a natural number m such that for all $k \geq 1$,* $s_k(T) \leq 2^{mk^2}$.

Our proof of this Theorem depends on a sequence of lemmas. Lemmas 1.3 and 1.4(i) below are well-known.

LEMMA 1.2: *For all* $k > 1$,
(i) $s_k(\mathcal{M}) \leq \max_{|A|=k-1} s_1(A, M) \cdot s_{k-1}(\mathcal{M})$; *and*
(ii) $s_k(\mathcal{M}) \leq s_1(\mathcal{M}) \cdot \Pi_{i=1}^{k-1} \max_{|A|=i} s_1(A, M)$.

Proof: (i) Let $\bar{a}_1, \cdots, \bar{a}_{s_{k-1}(M)}$ be realizations of the distinct $(k-1)$-types of M and for each $1 \leq i \leq s_{k-1}(\mathcal{M})$, let $b_i^j, 1 \leq j \leq s_1(\mathrm{rng}(\bar{a}_i), \mathcal{M})$ be realizations of the distinct 1-types over $\bar{a}_i$ in M. Then

$$\{\bar{a}_i\hat{\ }b_i^j : 1 \leq i \leq s_{k-1}(\mathcal{M}), 1 \leq j \leq s_1(M, \mathrm{rng}(\bar{a}_i))\}$$

is a complete set of realizations of the distinct k-types of $\mathcal{M}$. To see this, let $\bar{c} = (c_1, \cdots, c_k) \in M^k$. There is an automorphism f of $\mathcal{M}$ taking $(c_1, \ldots, c_{k-1})$ onto $\bar{a}_i$ for some i and an automorphism g of $\mathcal{M}$ fixing $\bar{a}_i$ and taking $f(c_k)$ to some b_i^j. The composition $g \circ f$ maps $\bar{c}$ to $\bar{a}_i\hat{\ }b_i^j$.

(ii) is proved by induction on k, using (i) for the induction step. ∎

LEMMA 1.3: *Let E be a 0-definable equivalence relation on M^n and let $\mathcal{M}^*$ be the E-extension of $\mathcal{M}$. Let L^* be the language of $\mathcal{M}^*$.*
(i) *The restriction to M of each automorphism of $\mathcal{M}^*$ is an automorphism of $\mathcal{M}$, and each automorphism of $\mathcal{M}$ extends uniquely to an automorphism of $\mathcal{M}^*$.*
(ii) *Let $A \subseteq M^k, 1 \leq k \leq \omega$. Then A is definable in $\mathcal{M}^*$ iff A is definable in $\mathcal{M}$ and the rank of A is the same whether it is computed in $\mathcal{M}$ or in $\mathcal{M}^*$.*
(iii) *$\mathcal{M}^*$ is ω-categorical and ω-stable.*

LEMMA 1.4: (i) *Let $A \subseteq M$ be nonempty and definable. If $n \in \omega$ and $B \subseteq A^n$, then B is definable in $\mathcal{M}$ iff B is definable in $\mathcal{M}|A$. Also,* (rk, deg)(B) *is the same whether it is computed in $\mathcal{M}$ or in $\mathcal{M}|A$.*
(ii) *If $A \subseteq M$ is definable, then for all $n \in \omega$ and all $\bar{a}, \bar{b} \in A^n$, if $\mathrm{tp}_{\mathcal{M}|A}(\bar{a}) = \mathrm{tp}_{\mathcal{M}|A}(\bar{b})$ then $\mathrm{tp}_{\mathcal{M}}(\bar{a}) = \mathrm{tp}_{\mathcal{M}}(\bar{b})$. If A is 0-definable, the converse is also true.*
(iii) *Suppose $A \subseteq M$ is 0-definable, E is a 0-definable equivalence relation on A^n, and $\mathcal{P}^*$ is the E-extension of $\mathcal{M}|A$. Then there is a 0-definable equivalence relation E' on M^n and an E'-extension $\mathcal{M}^*$ of $\mathcal{M}$ such that in $\mathcal{M}^*|(A \cup A^n/E)$ and $\mathcal{P}^*$ exactly the same relations are 0-definable.*

Proof: (i) This is Lemma 1.4 of [CHL].

(ii) If $\mathrm{tp}_{\mathcal{M}}(\bar{a}) \neq \mathrm{t\ p}_{\mathcal{M}}(\bar{b})$, then there is some 0-definable relation R on M such that R holds of $\bar{a}$ but not of $\bar{b}$. Hence $\mathrm{tp}_{\mathcal{M}|A}(\bar{a}) \neq \mathrm{tp}_{\mathcal{M}|A}(\bar{b})$. If $\mathrm{tp}_{\mathcal{M}}(\bar{a}) = \mathrm{tp}_{\mathcal{M}}(\bar{b})$, then there is an automorphism f of $\mathcal{M}$ taking $\bar{a}$ to $\bar{b}$. Assuming A is 0-definable,

f fixes A (setwise) and from the definition of $\mathcal{M}|A$ it follows that $f|A$ is an automorphism of $\mathcal{M}|A$ and thus $\mathrm{tp}_{\mathcal{M}|A}(\bar{a}) = \mathrm{tp}_{\mathcal{M}|A}(\bar{b})$.

(iii) Since A is 0-definable, E can be extended to a 0-definable equivalence relation E' on M^n by putting all elements of $M^n - A^n$ into one class. Let $\mathcal{M}^*$ be the E'-extension of $\mathcal{M}$. Then exactly the same relations are 0-definable in $\mathcal{P}^*$ and $\mathcal{M}^*|(A \cup A^n/E')$. ∎

LEMMA 1.5: *If $\mathcal{M}$ is strongly minimal, there is a $q < \omega$ such that for all non-empty finite subsets $A \subseteq M$, $|\mathrm{acl}(A)| \leq q^{|A|}$.*

Proof: First assume that $\mathcal{M}$ is strictly minimal. Consider the structure $H_{\mathcal{M}} = \, < M, R_i^{H_{\mathcal{M}}} >_{i \in \omega}$, where

$$R_i^{H_{\mathcal{M}}} = \{(a_1, \ldots, a_i, b) : b \in \mathrm{acl}(a_1, \ldots, a_i)\}.$$

By [CHL, 2.1], $H_{\mathcal{M}} \cong H_{\mathcal{P}}$ where $\mathrm{Aut}(\mathcal{P}) = \mathrm{AG}(\omega, q), \mathrm{PG}(\omega, q)$ or $\mathrm{Sym}(P)$. In the first two cases, the algebraic closure of a set A is the affine (resp. projective) span of A and thus $|\mathrm{acl}(A)| \leq q^{|A|}$, where q is the characteristic of the associated field. In the last case, $\mathrm{acl}(A) = A$. Now any isomorphism f of $H_{\mathcal{M}}$ onto $H_{\mathcal{P}}$ must take $\mathrm{acl}(A)$ onto $\mathrm{acl}(f[A])$ in $\mathcal{P}$. Therefore, since the result holds in $\mathcal{P}$ it must hold in $\mathcal{M}$ as well.

Returning to the general case, define E on M by $E(a, b)$ iff $\mathrm{acl}(a) = \mathrm{acl}(b)$. Since $\mathrm{rk}(\mathcal{M}) = 1$, for $a, b \in M - \mathrm{acl}(\emptyset), E(a, b)$ holds iff $a \in \mathrm{acl}(b)$ iff $b \in \mathrm{acl}(a)$; otherwise $E(a, b)$ holds iff $\{a, b\} \subseteq \mathrm{acl}(\emptyset)$. Therefore there is some $f < w$ such that for all $a \in M$, $|a/E| < f$. Let $\mathcal{M}^*$ be the E-extension of $\mathcal{M}$. We verify that $\mathcal{M}^*|(M - \mathrm{acl}(\emptyset))/E$ is strictly minimal. Suppose $S \subseteq (M - \mathrm{acl}(\emptyset))/E$ is infinite, coinfinite and definable in $\mathcal{M}^*|(M - \mathrm{acl}(\emptyset))/E$. By Lemma 1.4(i), S is definable in $\mathcal{M}^*$. Then $\cup S$ is an infinite, coinfinite subset of $M - \mathrm{acl}(\emptyset)$ definable in $\mathcal{M}^*$ and therefore in $\mathcal{M}$, by Lemma 1.3(ii). This contradicts the strong minimality of $\mathcal{M}$. To show strict minimality, suppose $a/E \in \mathrm{acl}(b/E)$, for some $a, b \in M - \mathrm{acl}(\emptyset)$, with algebraic closure computed in $\mathcal{M}^*|(M - \mathrm{acl}(\emptyset))/E$. By Lemma 1.4(ii), the same holds with algebraic closure computed in $\mathcal{M}$. Since each E-class is finite, Lemma 1.3(i) implies $a \in \mathrm{acl}(b)$ and thus $a/E = b/E$.

Let $A \subseteq M$ be finite and non-empty and let $A' = A - \mathrm{acl}(\emptyset)$. Assume that $A' \neq \emptyset$ (otherwise $\mathrm{acl}(A) = \mathrm{acl}(\emptyset)$). We show that

$$\mathrm{acl}(A) = \mathrm{acl}(A') = \cup\{b/E \in (M - \mathrm{acl}(\emptyset))/E : b/E \in \mathrm{acl}(A'/E)\} \cup \mathrm{acl}(\emptyset).$$

This is sufficient, since by the argument above, there is a $q < \omega$ such that $|\mathrm{acl}(A'/E)| \leq q^{|A'/E|} \leq q^{|A|}$. Therefore

$$|\cup \{b/E \in (M - \mathrm{acl}(\emptyset))/E : b/E \in \mathrm{acl}(A'/E)\} \cup \mathrm{acl}(\emptyset)| \leq f(q^{|A|} + 1)$$

(with f and q independent of A), so q can be chosen large enough to give the result.

Suppose $b \in \mathrm{acl}(A') - \mathrm{acl}(\emptyset)$. If $b/E \notin \mathrm{acl}(A'/E)$, there are distinct $\{b_i : i < \omega\} \subseteq M - \mathrm{acl}(\emptyset)$ and automorphisms $\{\hat{f}_i : i < \omega\}$ of $\mathcal{M}$ such that for each $i, \hat{f}_i$ fixes A'/E pointwise and $\hat{f}_i(b/E) = b_i/E$. For each $i, f_i = \hat{f}_i|M$ is an automorphism of $\mathcal{M}$ and we may assume that $f_i(b) = b_i$. Write $A' = \{a_1, \ldots, a_k\}$. Since $| \cup A'/E| < \omega$, there must be some $c_1, \ldots, c_k \in \cup A'/E$ such that for infinitely many $i, f_i(a_j) = c_j$ for all $1 \leq j \leq k$. This implies that for infinitely many $i, (a_1, \ldots, a_k, b)$ has the same type as $(c_1, \ldots, c_k, b_i)$, which contradicts $b \in \mathrm{acl}(a_1, \ldots, a_k)$. ∎

LEMMA 1.6: *Let $\mathcal{M}$ be a countable ω-categorical, ω-stable structure and let N be an infinite definable subset of M. Then there is a natural number q such that for all finite non-empty subsets $A \subseteq M$, the number of 1-types over A realized by elements of N in the structure $\mathcal{M}$ is less than or equal to $q^{|A|}$.*

Proof: We argue by induction on $\mathrm{rk}(N)$.

Base ($\mathrm{rk}(N) = 1$): To begin with, we assume

(*) N is an atom of $\mathcal{M}$.

By the finite equivalence relation theorem [CHL, 1.6] there is a 0-definable equivalence relation F on N such that $|N/F| < \omega$ and each of the F-classes is finite or strongly minimal. From (*) it follows that there are no finite F-classes. Let t be the number of F-classes, let C be an F-class and let $A \subseteq M$ be finite. Since C is strongly minimal, only one non-algebraic 1-type over A is realized in C. Let $S = \mathrm{acl}(A) \cap C$. Since (*) implies $\mathrm{acl}(\emptyset) \cap N = \emptyset$, $S = \{c \in C : \mathrm{tp}(c/A)$ forks over $\emptyset\}$. Write $A = \{a_1, \ldots, a_{|A|}\}$ and let $\bar{a} = (a_1, \ldots, a_{|A|})$. By [CHL, 1.3(i)] there are at most $\mathrm{rk}(\bar{a}/\emptyset)$ algebraically independent elements in S. By [CHL, 1.2], $\mathrm{rk}(\bar{a}/\emptyset) \leq \mathrm{rk}(M^{|A|}) = |A|\mathrm{rk}(M)$. Let $m = \mathrm{rk}(M)$ and let $B \subseteq S$ satisfy $|B| \leq m|A|$ and $\mathrm{acl}(B) \cap C = S$. By Lemma 1.4(ii), $\mathrm{acl}(B) \cap C$ is contained in the algebraic closure of B computed in $\mathcal{M}|C$. By Lemma 1.4(i), $\mathcal{M}|C$ is strongly minimal. Thus by Lemma 1.5, $|S| \leq q^{m|A|}$ for some $q \in \omega$, independent of A. Hence there are at most $q^{m|A|} + 1$ 1-types (of $\mathcal{M}$) over A realized in C and thus in all of N there are at most $t(r^{m|A|}+1)$ realized, where r is the maximum of the q's obtained for the F-classes. Since t, r and m are independent of A, q can be chosen large enough to give the result.

Now remove the assumption (*). If N is definable over $\bar{a}$, then it is the union of a finite number of atoms of $(\mathcal{M}, \bar{a})$. The previous argument applies to the infinite atoms, and since there are only finitely many elements in the finite atoms, the result follows for N and $(\mathcal{M}, \bar{a})$, which implies the result for N and $\mathcal{M}$, since $\mathrm{Aut}(\mathcal{M}, \bar{a}) \leq \mathrm{Aut}(\mathcal{M})$.

INDUCTIVE STEP: Let $n = \mathrm{rk}(N) > 1$. We first show that we can reduce to the case in which N is an atom of $\mathcal{M}$ and $\deg(N) = 1$. Expand to $\mathcal{P} = (\mathcal{M}, \bar{a})$ where N is definable over $\bar{a}$. As in the base case, the result for $\mathcal{P}$ and N will imply the result for N and $\mathcal{M}$. Again by the finite equivalence relation theorem there is a 0-definable equivalence relation F on N such that $|N/F| < \omega$ and each F-class has degree 1 or rank less than n. The result holds by induction for F-classes of positive rank less than n and the union of the finite F-classes is finite. Therefore it suffices to prove the result for an infinite F-class G with $(\mathrm{rk},\deg)(G) = (n, 1)$. To make the final reduction, introduce one more parameter, say g, to define G in P and decompose G into atoms of $(\mathcal{P}, g)$.

Let $N \subseteq M$ be as above and let $A \subseteq M$ be finite and non-empty. By the proof of the Coordinatization Theorem [CHL, 4.1], there is a 0-definable equivalence relation E on N^k for some k, an E-extension $\mathcal{N}^*$ of $\mathcal{M}|N$, a rank one atom $C \subseteq N^*$ and a formula $\varphi(x, y)$ in the language of $\mathcal{N}^*$ such that each element of N belongs to at least one set $\varphi(x, c)^{\mathcal{N}^*}, c \in C$, and for all $c \in C, \mathrm{rk}(\varphi(x, c)) = n-1$ (computed in $\mathcal{N}^*$). Since N is 0-definable in $\mathcal{M}$, by Lemma 1.4(iii) there is an E' extending E to M^k and an E'-extension $\mathcal{M}^*$ of M such that $\mathcal{N}^*$ has the same 0-definable relations as $\mathcal{M}^*|(N \cup N^k/E)$. By Lemma 1.4(i), C is definable in $\mathcal{M}^*$ and has rank 1. By induction, the number of 1-types over A (of $\mathcal{M}^*$) realized in C is less than or equal to $q^{|A|}$ for some q independent of A. Thus among the sets $\varphi(x, c)^{\mathcal{N}^*}$ there are at most $q^{|A|}$ conjugacy classes over A. Since by Lemma 1.4(i) each of the sets $\varphi(x, c)^{\mathcal{N}^*}$ is definable of rank $n-1$ in $\mathcal{M}^*$, by induction for each $c \in C$ there is q_c such that there are at most $q_c^{|A|}$ 1-types (of $\mathcal{M}^*$) realized in $\varphi(x, c)^{\mathcal{N}^*}$. Since C is an atom, q_c is independent of c. Thus the number of 1-types (of $\mathcal{M}^*$) over A realized in N is less than or equal to $(q_c q)^{|A|}$. By Lemma 1.3(i) this bound is valid in $\mathcal{M}$ as well. ∎

Proof of Theorem 1.1: Let $M \models T$. Applying Lemma 1.6 with $N = M$ together with Lemma 1.2(ii) gives immediately for all $k > 1$,

$$s_k(\mathcal{M}) \leq s_1(\mathcal{M}) \cdot \prod_{i=1}^{k-1} q^i < s_1(\mathcal{M}) \cdot q^{k^2}, \quad \text{for some } q \in \omega.$$

Thus choosing $m > \log_2(s_1(\mathcal{M}) \cdot q)$, the result follows. ∎

Remarks: (1) If T is the (strictly minimal) theory of an infinite dimensional affine space over a finite field, then by [M, 3.4] $s_k(T) \geq q^{p(k)}$ for all k, where q is the characteristic of the field and p is a quadratic polynomial with leading term $k^2/4$. Thus the bound in Theorem 1.1 is sharp, since for each m there is a (prime) q such that for any quadratic $p(k) = k^2/4 + bk + c, q^{p(k)}$ eventually dominates 2^{mk^2}.

(2) Chris Laskowski and Udi Hrushovski have shown (personal communication) that by modifying the construction described in [H], one can obtain stable, but not ω-stable theories whose sequences $s_k(T)$ grow arbitrarily fast. Thus we cannot weaken the stability requirement in Theorem 1.1. ∎

2. Algebraic Closure

In proving Theorem 1.1, we bounded the number of 1-types over a k-element subset of $\mathcal{M}$ (Lemma 1.6). To get the induction started in the proof of Lemma 1.6, we needed the fact that we could bound the size of the algebraic closure of a k-element subset of a strongly minimal structure $\mathcal{M}$ (Lemma 1.5). This followed easily from the Classification Theorem [CHL, 2.1] for strictly minimal geometries. We now show, using a similar inductive argument, that Lemma 1.5 holds for all ω-categorical, ω-stable structures.

THEOREM 2.1: *Let T be ω-categorical and ω-stable and let $\mathcal{M} \models T$. Then there is $q < \omega$ such that for all non-empty finite subsets $A \subseteq M, |\mathrm{acl}(A)| \leq q^{|A|}$.*

By considering $N = M$, the Theorem follows directly from:

LEMMA 2.2: *Let $\mathcal{M}$ be a countable ω-categorical, ω-stable structure and let N be an infinite definable subset of M. Then there is a natural number q such that for all finite non-empty subsets $A \subseteq M, |\mathrm{acl}(A) \cap N| \leq q^{|A|}$.*

Proof: As in the proof of Lemma 1.6, we argue by induction on $\mathrm{rk}(N)$. The base case, $\mathrm{rk}(N) = 1$, is established in the proof of Lemma 1.6. So assume $\mathrm{rk}(N) > 1$.

The reduction to the case in which N is a degree 1 atom works as before. Now let C, N^* and the coordinatizing sets $\varphi(x,c)^{\mathcal{N}^*}, c \in C$ be defined as in the proof of Lemma 1.6. By induction, it suffices to show:

$$(*) \qquad \mathrm{acl}(A) \cap N \subseteq \{\mathrm{acl}(A) \cap \varphi(x,c)^{\mathcal{N}^*} : c \in \mathrm{acl}(A) \cap C\}.$$

Let $a \in \mathrm{acl}(A) \cap N$. First note that, though we did not need this fact in the proof of Lemma 1.6, the sets $\varphi(x,c)^{\mathcal{N}^*}$ obtained from (the proof of) the [CHL] Coordinatization Theorem have the property that each $a \in N$ belongs to a positive but finite number of the sets $\varphi(x,c)^{\mathcal{N}^*}, c \in C$. Therefore $a \in \varphi(x,c)^{\mathcal{N}^*}$, some $c \in C$, with $c \in \mathrm{acl}(a)$ (computed in $\mathcal{N}^*$, using the fact that C is 0-definable). But if $c \in \mathrm{acl}(a)$ and $a \in \mathrm{acl}(A)$, then $c \in \mathrm{acl}(A)$ and $(*)$ follows. ∎

3. Theories with Only Indiscernible Strictly Minimal Sets

In this section, we improve the bound in Theorem 1.1 for a special subclass of ω-categorical, ω-stable theories.

THEOREM 3.1: *Let T be an ω-stable, ω-categorical theory and let $\mathcal{M}$ be the countable model of T. Suppose that every strictly minimal set definable in any extension by definitions of $\mathcal{M}$ is indiscernible. Then for some $c \in \omega, s_k(T) \leq c^k(k!)^{\mathrm{rk}(M)}$ for all $k \geq 1$.*

Considering $N = M$ and using Lemma 1.2, it suffices to establish:

LEMMA 3.2: *Let $\mathcal{M}$ be as in Theorem 3.1 and let N be an infinite definable subset of M. Then there is $c \in \omega$ such that for all finite non-empty subsets $A \subseteq M$, the number of 1-types over A realized by elements of N in the structure $\mathcal{M}$ is less than or equal to $c|A|^{\mathrm{rk}(N)}$.*

Proof: Once again, we prove this by induction on $\mathrm{rk}(N)$. The base case works essentially as in the proof of Lemma 1.6. First, the proof of Lemma 1.5 (with minor modifications) shows that if $\mathcal{M}$ is strongly minimal and has only indiscernible strictly minimal sets attached, then for all $A \subseteq M, |A| < \omega, |\mathrm{acl}(A)| < f|A|$, with $f \in \omega$ as defined in the proof. Now follow the proof of Lemma 1.6. Assuming that N is an atom of $\mathcal{M}$, decompose N into strongly minimal components $C_0, \ldots, C_m$ and see using [CHL 1.3(i),1.2] that for any finite $A \subseteq M$, $\mathrm{acl}(A) \cap C_i$ has a basis of cardinality $\leq |A| \cdot \mathrm{rk}(M)$ for each $i \leq m$. Thus $|\mathrm{acl}(A) \cap C_i| \leq f_i \cdot |A| \cdot \mathrm{rk}(M)$ each i, and

$$|\mathrm{acl}(A) \cap N| \leq m \cdot \max_{i \leq m}(f_i) \cdot |A| \cdot \mathrm{rk}(M).$$

Removing the assumption that N is an atom works as before, using the fact that $(\mathcal{M}, \bar{a})$ still satisfies the hypotheses of the Theorem.

In the inductive step, the preliminary reduction works as before. Then with the sets C and $\varphi(x,c)^{\mathcal{N}^*}$ defined as in the proof of Lemma 1.6, we have by induction for each finite $A \subseteq M$,

$$|S_1(A,C)| \leq f \cdot |A| \quad \text{and} \quad |S_1(A, \varphi(x,c)^{\mathcal{N}^*})| \leq d \cdot |A|^{n-1},$$

all $c \in C$, for some $f, d \in \omega$. Then $|S_1(A,N)| \leq f \cdot |A| \cdot d \cdot |A|^{n-1} = fd \cdot |A|^n$. ∎

Questions: (1) Do either of the two Theorems 1.1, 2.1 have (partial) converses? That is, does "slow" growth in either $s_k(T)$ or $a_k(T) = \max_{|A|=k} |\mathrm{acl}(A)|$ imply that a stable ω-categorical theory is ω-stable? The hypothesis of stability is certainly necessary here. Consider, e.g., $T = \mathrm{Th}(\mathbb{Q}, \leq)$. Then $s_k(T) < (2k)!$.

(2) For stable, ω-categorical T can we bound $s_k(T)$ explicitly in terms of $a_k(T)$? This is true for the theories constructed in [H], and our arguments suggest that it should be possible at least in the ω-stable case. ∎

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